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Cantor–Bernstein Theorem for Pseudo-BCK-Algebras

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Abstract We prove that if A and B are orthogonally σ-complete commutative pseudo-BCK-algebras such that A is isomorphic to a direct factor in B, and also B is isomorphic to a direct factor in A, then A and B are isomorphic. As a consequence we obtain previously known results for MV-algebras (by De Simone, Mundici and Navara), pseudo-MV-algebras (by Jakubík) and lattice-ordered groups (again by Jakubík).

Keywords · Commutative pseudo-BCK-algebra · Orthogonal σ-completeness · Direct factor · Cantor–Bernstein theorem

1 Introduction and the main result

A *pseudo-BCK-algebra* [5] is a structure $(A, \leqslant, \oslash, \odot, 0)$ where $(A, \leqslant)$ is a poset with a least element 0, and $\oslash$, $\odot$ are binary operations on A such that, for all $x, y, z \in A$, we have

$$(z \oslash y) \odot (z \oslash x) \leqslant x \oslash y, \qquad (z \odot y) \oslash (z \odot x) \leqslant x \odot y, \tag{1.1}$$

$$x \oslash (x \odot y) \leqslant y, \qquad x \odot (x \oslash y) \leqslant y, \tag{1.2}$$

$$x \leqslant y \quad \text{iff} \quad x \oslash y = 0 \quad \text{iff} \quad x \odot y = 0. \tag{1.3}$$

We say that a pseudo-BCK-algebra $(A, \leqslant, \oslash, \odot, 0)$ is *commutative* if it satisfies the identities

$$x \oslash (x \odot y) = y \oslash (y \odot x), \qquad x \odot (x \oslash y) = y \odot (y \oslash x). \tag{1.4}$$

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It is not hard to show that the poset $(A, \leqslant)$ then is a meet-semilattice in which the infimum $x \wedge y$ is given by $x \wedge y = x \varobslash (x \oslash y) = x \oslash (x \varobslash y)$. Further, a commutative pseudo-BCK-algebra $(A, \leqslant, \oslash, \varobslash, 0)$ is called *orthogonally σ-complete* if every countable pairwise orthogonal subset $X \subseteq A$ (i.e., $x \wedge y = 0$ for all $x, y \in X$, $x \neq y$) has a supremum in A.

In the present paper we prove the following Cantor–Bernstein type theorem:

Theorem A *Let A and B be orthogonally σ-complete commutative pseudo-BCK-algebras and assume that*

$$A \cong B \times C_1 \quad \textit{and} \quad B \cong A \times C_2$$

for some pseudo-BCK-algebras C_1, C_2. Then $A \cong B$.

If $(A, \vee, \wedge, ', 0, 1)$ is a boolean algebra then $(A, \leqslant, \oslash, \varobslash, 0)$, where $x \oslash y = x \varobslash y := x \wedge y'$, is a commutative pseudo-BCK-algebra, and hence Theorem A is a generalization of the following known result by Sikorski and Tarski: *If A and B are σ-complete boolean algebras such that $A \cong [0, b] \subseteq B$ and $B \cong [0, a] \subseteq A$ for some $a \in A$, $b \in B$, then $A \cong B$.*

Since pseudo-MV-algebras and positive cones of ℓ-groups can be regarded as special cases of commutative pseudo-BCK-algebras (see Example 2.2), the theorem is applicable to MV-algebras and pseudo-MV-algebras as well as to ℓ-groups (in Sect. 5 we obtain the main results from [15], [10] and [9] as easy consequences of Theorem A).

Other Cantor–Bernstein like theorems extending the aforementioned result for boolean algebras were proved in [16] for orthomodular lattices, in [11] for effect algebras and in [1] for pseudo-effect algebras. These theorems, however, are incomparable with Theorem A, because roughly speaking the intersection of orthomodular lattices and (pseudo-)effect algebras with commutative pseudo-BCK-algebras are boolean algebras and (pseudo-)MV-algebras, respectively. An abstract version for algebras that have an underlying bounded lattice order (satisfying certain additional conditions) can be found in [3]. Likewise this result is incomparable with Theorem A since commutative pseudo-BCK-algebras in general are meet-semilattices without upper bound, and bounded commutative pseudo-BCK-algebras are equivalent to pseudo-MV-algebras (cf. Example 2.2(b)).

2 Pseudo-BCK-Algebras

Pseudo-BCK-algebras were introduced by Georgescu and Iorgulescu [5] and generalize well-known BCK-algebras in the sense that if the binary operations $\oslash$ and $\varobslash$ coincide then the resulting structure becomes a BCK-algebra. The definition we have used at the beginning is essentially the original one, but it is easily seen that pseudo-BCK-algebras can be treated as algebras $(A, \oslash, \varobslash, 0)$ of type $\langle 2, 2, 0\rangle$. Indeed, if $(A, \leqslant, \oslash, \varobslash, 0)$ is pseudo-BCK-algebra then the algebra $(A, \oslash, \varobslash, 0)$ satisfies the following identities and quasi-identity:

$$[(z \oslash y) \varobslash (z \oslash x)] \varobslash (x \oslash y) = 0, \tag{2.1}$$

$$[(z \varobslash y) \oslash (z \varobslash x)] \oslash (x \varobslash y) = 0, \tag{2.2}$$

$$x \oslash 0 = x, \tag{2.3}$$

$$x \varobslash 0 = x, \tag{2.4}$$

$$0 \oslash x = 0, \tag{2.5}$$

$$x \oslash y = 0 \quad \& \quad y \oslash x = 0 \quad \Rightarrow \quad x = y, \tag{2.6}$$

and on the other hand, on every algebra $(A, \oslash, \otimes, 0)$ satisfying (2.1—2.6) one can define a partial order $\leqslant$ by setting $x \leqslant y$ iff $x \oslash y = 0$ (which is equivalent to $x \otimes y = 0$) so that $(A, \leqslant, \oslash, \otimes, 0)$ is a pseudo-BCK-algebra.

Accordingly, by a pseudo-BCK-algebra we shall mean an algebra $(A, \oslash, \otimes, 0)$ that fulfills (2.1—2.6). A *bounded* pseudo-BCK-algebra is an algebra $(A, \oslash, \otimes, 0, 1)$, where $(A, \oslash, \otimes, 0)$ is a pseudo-BCK-algebra with a greatest element 1.

Since BCK-algebras coincide with those pseudo-BCK-algebras for which $\oslash = \otimes$, and BCK-algebras are not closed under homomorphic images, it follows that the class of all pseudo-BCK-algebras is a proper quasi-variety.

We have already mentioned that every boolean algebra can be made into a bounded commutative (pseudo-)BCK-algebra. Moreover, it is known that an arbitrary poset $(P, \leqslant)$ with least element 0 becomes a (pseudo-)BCK-algebra if we define $x \oslash y = x \otimes y := 0$ if $x \leqslant y$, and $x \oslash y = x \otimes y := x$ otherwise.

In general, pseudo-BCK-algebras arise as subreducts of dually integral dually residuated partially ordered monoids:

Example 2.1 A *dually integral dually residuated partially ordered monoid* (*po-monoid* for short) is a structure $(M, \leqslant, \oplus, \oslash, \otimes, 0)$ where $(M, \leqslant)$ is a partially ordered set, $(M, \oplus, 0)$ is a monoid whose identity 0 is the least element of $(M, \leqslant)$, and

$$a \oplus b \geqslant c \quad \text{iff} \quad a \geqslant c \oslash b \quad \text{iff} \quad b \geqslant c \otimes a$$

for all $a, b, c \in M$. A straightforward verification yields that $(M, \oslash, \otimes, 0)$ is a pseudo-BCK-algebra. Conversely, by [12], every pseudo-BCK-algebra is obtained as a subalgebra of the reduct $(M, \oslash, \otimes, 0)$ of a suitable dually integral dually residuated po-monoid $(M, \leqslant, \oplus, \oslash, \otimes, 0)$.

In addition to (1.1—1.3) and (2.1—2.6), pseudo-BCK-algebras satisfy the following easily derivable properties (see [5]):

$$x \oslash x = x \otimes x = 0, \qquad 0 \oslash x = 0 \otimes x = 0, \tag{2.7}$$

$$x \leqslant y \quad \Rightarrow \quad x \oslash z \leqslant y \oslash z \quad \& \quad x \otimes z \leqslant y \otimes z, \tag{2.8}$$

$$x \leqslant y \quad \Rightarrow \quad z \oslash y \leqslant z \oslash x \quad \& \quad z \otimes y \leqslant z \otimes x, \tag{2.9}$$

$$(x \oslash y) \otimes z = (x \otimes z) \oslash y, \tag{2.10}$$

$$x \oslash y \leqslant z \quad \text{iff} \quad x \otimes z \leqslant y, \tag{2.11}$$

$$x \oslash y \leqslant x, \qquad x \otimes y \leqslant x, \tag{2.12}$$

$$(x \oslash z) \oslash (y \oslash z) \leqslant x \oslash y, \qquad (x \otimes z) \otimes (y \otimes z) \leqslant x \otimes y, \tag{2.13}$$

$$x \oslash (x \otimes (x \oslash y)) = x \oslash y, \qquad x \otimes (x \oslash (x \otimes y)) = x \otimes y. \tag{2.14}$$

Moreover, if $\bigwedge_{i \in I} y_i$ exists, then so does $\bigvee_{i \in I} (x \oslash y_i)$ and

$$x \oslash \left(\bigwedge_{i \in I} y_i \right) = \bigvee_{i \in I} (x \oslash y_i), \tag{2.15}$$

and the same holds for $\otimes$; in particular, if $x \wedge y$ exists then

$$x \oslash (x \wedge y) = x \oslash y \quad \text{and} \quad x \otimes (x \wedge y) = x \otimes y. \tag{2.16}$$

Since Theorem A is concerned exclusively with commutative pseudo-BCK-algebras, we focus our attention on them. There are two especially relevant examples:

Example 2.2 (a) Let $(G, +, -, 0, \vee, \wedge)$ be an ℓ-group and $G^+ = \{x \in G : x \geqslant 0\}$ its positive cone. If we put

$$x \oslash y := (x - y) \vee 0 \quad \text{and} \quad x \varobslash y := (-y + x) \vee 0,$$

then $(G^+, \oslash, \varobslash, 0)$ is a commutative pseudo-BCK-algebra. (Note that $(G^+, \leqslant, +, \oslash, \varobslash, 0)$ is a dually integral dually residuated po-monoid.) We emphasize that *not* all commutative pseudo-BCK-algebras arise in this way as subalgebras of $(G^+, \oslash, \varobslash, 0)$ (see [2]).

(b) A *pseudo-MV-algebra* $(M, \oplus, ^-, ^\sim, 0, 1)$ is a monoid $(M, \oplus, 0)$ endowed with a constant 1 and two supplementary unary operations satisfying the equations:

$$\begin{aligned}
&x \oplus 1 = 1 = 1 \oplus x,\\
&1^- = 0 = 1^\sim,\\
&(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-,\\
&x \oplus (y \odot x^\sim) = y \oplus (x \odot y^\sim) = (y^- \odot x) \oplus y = (x^- \odot y) \oplus x,\\
&(x^- \oplus y) \odot x = y \odot (x \oplus y^\sim),\\
&x^{-\sim} = x,
\end{aligned}$$

where the binary operation $\odot$ is defined by $x \odot y = (x^- \oplus y^-)^\sim$. Pseudo-MV-algebras were established in [4] and independently in [14] as a non-commutative extension of MV-algebras; indeed, if the monoid $(M, \oplus, 0)$ is commutative, then $^-$ and $^\sim$ coincide and $(M, \oplus, ^-, 0, 1)$ is an MV-algebra.

Let $(M, \oplus, ^-, ^\sim, 0, 1)$ be a pseudo-MV-algebra and define

$$x \oslash y := (y \oplus x^\sim)^- \quad \text{and} \quad x \varobslash y := (x^- \oplus y)^\sim.$$

Then $(M, \oslash, \varobslash, 0, 1)$ is a bounded commutative pseudo-BCK-algebra (and $(M, \leqslant, \oplus, \oslash, \varobslash, 0)$ is a dually integral dually residuated po-monoid, where $\leqslant$ denoted the natural order of M defined by $x \leqslant y$ iff $x^- \oplus y = 1$). By [5], bounded commutative pseudo-BCK-algebras are even termwise equivalent to pseudo-MV-algebras; the equivalence is given by the stipulation

$$\begin{aligned}
&x \oplus y := 1 \oslash [(1 \varobslash x) \varobslash y] = 1 \varobslash [(1 \oslash y) \oslash x],\\
&x^- := 1 \oslash x \quad \& \quad x^\sim := 1 \varobslash x.
\end{aligned}$$

Another important observation is that commutative pseudo-BCK-algebras form an equational class [13]:

Lemma 2.3 *An algebra $(A, \oslash, \varobslash, 0)$ of type $\langle 2, 2, 0\rangle$ is a commutative pseudo-BCK-algebra if and only if it satisfies* (2.7) *and* (2.10) *together with the identities*

$$x \oslash (x \varobslash y) = y \oslash (y \varobslash x) = x \varobslash (x \oslash y) = y \varobslash (y \oslash x).$$

It is clear by (2.12) that every interval $[0,a]$ of any commutative pseudo-BCK-algebra is a bounded commutative pseudo-BCK-algebra, hence a pseudo-MV-algebra. More precisely, if we are given a commutative pseudo-BCK-algebra $(A,\oslash,\odot,0)$ and $0<a\in A$, and define

$$x\oplus_a y := a\oslash[(a\odot x)\odot y] = a\odot[(a\oslash y)\oslash x],$$
$$x^{-a} := a\oslash x \quad \& \quad x^{\sim a} := a\odot x,$$

then $([0,a],\oplus_a,{}^{-a},{}^{\sim a},0,a)$ is a pseudo-MV-algebra. Consequently, although $(A,\leqslant)$ is a meet-semilattice that need not be a lattice, the segment $([0,a],\leqslant)$ is a distributive lattice in which, for all $x,y\in[0,a]$, the supremum $x\vee_a y$ is expressed by the formulae

$$\begin{aligned} x\vee_a y &= (x^{\sim a}\wedge y^{\sim a})^{-a} = a\oslash[(a\odot x)\odot(y\odot x)] = x\oplus_a(y\odot x)\\ &= (x^{-a}\wedge y^{-a})^{\sim a} = a\odot[(a\oslash x)\oslash(y\oslash x)] = (y\oslash x)\oplus_a x. \end{aligned}$$

Lemma 2.4 *Let $(A,\oslash,\odot,0)$ be a commutative pseudo-BCK-algebra. If the suprema indicated on the left-hand side exist, then also the suprema and infima on the right-hand side exist and the following equalities hold*:

(a) $x\wedge\bigvee_{i\in I}y_i=\bigvee_{i\in I}(x\wedge y_i)$;
(b) $(\bigvee_{i\in I}x_i)\oslash y=\bigvee_{i\in I}(x_i\oslash y)$, *and the same for* $\odot$;
(c) $x\oslash(\bigvee_{i\in I}y_i)=\bigwedge_{i\in I}(x\oslash y_i)$, *and the same for* $\odot$.

Proof First of all note that (a), (b) and (c) are valid in pseudo-MV-algebras. The idea is to calculate the respective joins and meets in a suitable pseudo-MV-algebra $[0,a]$, $a\in A$.

(a) Put $a=\bigvee_{i\in I}y_i$. Since $x\wedge a\in[0,a]$ and $y_i\in[0,a]$ for all $i\in I$, we get $x\wedge\bigvee_{i\in I}y_i=(x\wedge a)\wedge\bigvee_{i\in I}y_i=\bigvee_{i\in I}(x\wedge a)\wedge y_i=\bigvee_{i\in I}x\wedge y_i$.

(b) With $a=\bigvee_{i\in I}x_i$ we have $(\bigvee_{i\in I}x_i)\oslash y=(\bigvee_{i\in I}x_i)\oslash(a\wedge y)=\bigvee_{i\in I}(x_i\oslash(a\wedge y))=\bigvee_{i\in I}((x_i\oslash a)\vee(x_i\oslash y))=\bigvee_{i\in I}(x_i\oslash y)$ as $x_i\oslash a=0$.

(c) We can use the interval $[0,x]$ now. Applying the property (a) we get $x\oslash(\bigvee_{i\in I}y_i)=x\oslash(x\wedge\bigvee_{i\in I}y_i)=x\oslash(\bigvee_{i\in I}(x\wedge y_i))=\bigwedge_{i\in I}(x\oslash(x\wedge y_i))=\bigwedge_{i\in I}(x\oslash y_i)$. □

3 Deductive Systems and Direct Factors

Throughout this paragraph we restrict ourselves to commutative pseudo-BCK-algebras; for more information about deductive systems in general pseudo-BCK-algebras we refer to [6].

Let $(A,\oslash,\odot,0)$ be a commutative pseudo-BCK-algebra. We call $D\subseteq A$ a *deductive system* of A if

(i) $0\in D$,
(ii) if $a\oslash b\in D$ and $b\in D$ then $a\in D$.

The second condition is equivalent to saying that if $a\odot b\in D$ and $b\in D$ then $a\in D$. For every $X\subseteq A$, there is the smallest deductive system $D(X)$ containing X; namely, $D(\emptyset)=\{0\}$, and

$$D(X)=\{a\in A:(\cdots(a\oslash x_1)\oslash\cdots)\oslash x_n=0 \text{ for some } x_i\in X, n\in\mathbb{N}\}$$

for $X\neq\emptyset$.

We use $\mathcal{DS}(A)$ to denote the set of all deductive systems of A. When ordered by inclusion, $\mathcal{DS}(A)$ forms an algebraic distributive lattice in which infima agree with set-theoretical intersections and, for every $\{X_i : i \in I\} \subseteq \mathcal{DS}(A)$, the supremum in $\mathcal{DS}(A)$ is $\bigvee_{i\in I} X_i = D(\bigcup_{i\in I} X_i)$.

The lattice $\mathcal{DS}(A)$ is pseudocomplemented; for each $X \in \mathcal{DS}(A)$,

$$\begin{aligned} X^\delta &= \{a \in A : a \wedge x = 0 \text{ for all } x \in X\} \\ &= \{a \in A : x \oslash a = x \text{ for all } x \in X\} \end{aligned}$$

is the pseudocomplement of X in $\mathcal{DS}(A)$. X^δ is called the *polar* of X.

Furthermore, we say that $J \in \mathcal{DS}(A)$ is a *compatible deductive system*, or an *ideal* of A if, for all $a, b \in A$, $a \oslash b \in J$ if and only if $a \varobslash b \in J$. Equivalently, J is an ideal iff $a \oslash (a \oslash x) \in J$ and $a \varobslash (a \varobslash x) \in J$ for all $a \in A$ and $x \in J$. The set of all ideals is denoted by $\mathcal{I}d(A)$; again, it is partially ordered by inclusion. The ideals of A correspond one-one to the congruences on A. That is, if $J \in \mathcal{I}d(A)$ then the relation Θ_J defined by

$$(a, b) \in \Theta_J \quad \text{iff} \quad a \oslash b \in J \text{ and } b \oslash a \in J$$

is a congruence on A such that $J = 0/\Theta_J = \{a \in A : (a, 0) \in \Theta_J\}$, and conversely, for any congruence Φ on A, its kernel $J = 0/\Phi$ is an ideal of A with $\Theta_J = \Phi$.

Let A be a commutative pseudo-BCK-algebra. It can be easily seen that if $\varphi: A \to A_1 \times A_2$ is an isomorphism of A onto the direct product $A_1 \times A_2$ of commutative pseudo-BCK-algebras A_1 and A_2, then both $\varphi^{-1}(A_1 \times \{0\})$ and $\varphi^{-1}(\{0\} \times A_2)$ are ideals of A. Moreover, the polar of $\varphi^{-1}(A_1 \times \{0\})$ is precisely $\varphi^{-1}(\{0\} \times A_2)$, and vice versa.

We shall say that a deductive system $X \in \mathcal{DS}(A)$ is a *direct factor* in A if $X = \varphi^{-1}(A_1 \times \{0\})$ or $X = \varphi^{-1}(\{0\} \times A_2)$ for some direct product decomposition $\varphi: A \to A_1 \times A_2$.

In this case, also X^δ is a direct factor and A is isomorphic to the direct product $X \times X^\delta$; we write $A = X \oplus X^\delta$ and say that A is the *direct sum* of X and X^δ.

We now describe the direct factors in orthogonally σ-complete commutative pseudo-BCK-algebras:

Lemma 3.1 *Let A be an orthogonally σ-complete commutative pseudo-BCK-algebra. For any $X \in \mathcal{DS}(A)$, the following are equivalent*:

(a) *X is a direct factor*;
(b) *every $a \in A$ can be written in the form $a = x \vee y$, where $x \in X$ and $y \in X^\delta$*;
(c) *for every $a \in A$, the set $X_a = \{x \in X : x \leqslant a\}$ has a greatest element.*

Proof (a) $\Rightarrow$ (b). Let $\varphi: A \to A_1 \times A_2$ be a direct product decomposition of A and assume that $X = \varphi^{-1}(A_1 \times \{0\})$. Then $X^\delta = \varphi^{-1}(\{0\} \times A_2)$. For every $a \in A$, if $\varphi(a) = (a_1, a_2)$ then $x = \varphi^{-1}(a_1, 0) \in X$ and $y = \varphi^{-1}(0, a_2) \in X^\delta$, hence $x \vee y$ exists in A (for x, y are orthogonal) and one readily sees that $\varphi(x \vee y) = \varphi(x) \vee \varphi(y) = (a_1, 0) \vee (0, a_2) = (a_1, a_2) = \varphi(a)$, so $x \vee y = a$.

(a) $\Rightarrow$ (c). With the above notation, it is obvious that $\varphi^{-1}(a_1, 0) \in X_a$. If $b \in X_a$ then $(b_1, 0) = \varphi(b) \leqslant \varphi(a) = (a_1, a_2)$, which yields $b = \varphi^{-1}(b_1, 0) \leqslant \varphi^{-1}(a_1, 0)$. Therefore $\varphi^{-1}(a_1, 0)$ is the greatest element of $X_a = \{x \in X : x \leqslant a\}$.

(b) $\Rightarrow$ (a). First we show the uniqueness of the expression $a = x \vee y$, where $x \in X$ and $y \in X^\delta$. Suppose that $a = x' \vee y'$ for some $x' \in X$, $y' \in X^\delta$. Then $x = x \oslash (x \wedge y) = x \oslash y = (x \vee y) \oslash y = (x' \vee y') \oslash y = (x' \oslash y) \vee (y' \oslash y) = (x' \oslash (x' \wedge y)) \vee (y' \oslash y) = x' \vee (y' \oslash y)$, thus $x \geqslant x'$. Analogously, $y = y' \vee (x' \oslash x)$, which along with $x \geqslant x'$ (i.e. $x' \oslash x = 0$) entails $y = y'$ and $x = x'$.

Consequently, the map $\psi\colon X \times X^\delta \to A$ defined by $\psi(x, y) = x \vee y$ is a bijection. For any $x, x' \in X$ and $y, y' \in X^\delta$, we have $\psi(x, y) \oslash \psi(x', y') = (x \vee y) \oslash (x' \vee y') = ((x \oslash x') \wedge (x \oslash y')) \vee ((y \oslash x') \wedge (y \oslash y')) = (x \wedge (x \oslash x')) \vee (y \wedge (y \oslash y')) = (x \oslash x') \vee (y \oslash y') = \psi((x, y) \oslash (x', y'))$, and similarly $\psi(x, y) \otimes \psi(x', y') = \psi((x, y) \otimes (x', y'))$. Thus ψ is an isomorphism. Therefore, $\varphi = \psi^{-1}\colon A \to X \times X^\delta$ is a direct product decomposition such that $\varphi^{-1}(X \times \{0\}) = X$ and $\varphi^{-1}(\{0\} \times X^\delta) = X^\delta$.

(c) $\Rightarrow$ (b). Given $a \in A$, let a_1 be the greatest element of X_a and put $a_2 = a \oslash a_1$. We have $((a \otimes (a_2 \oslash a_1)) \oslash a_1) \oslash a_1 = ((a \oslash a_1) \otimes (a_2 \oslash a_1)) \oslash a_1 = (a_2 \otimes (a_2 \oslash a_1)) \oslash a_1 = (a_1 \wedge a_2) \oslash a_1 = 0$; since $a_1 \in X$, it follows that $a \otimes (a_2 \oslash a_1) \in X$. Hence $a \otimes (a_2 \oslash a_1) \leqslant a_1$, because $a \otimes (a_2 \oslash a_1)$ is less than or equal to a and belongs to X. On the other hand, $a \otimes (a_2 \oslash a_1) \geqslant a \otimes a_2 = a \otimes (a \oslash a_1) = a \wedge a_1 = a_1$, so $a \otimes (a_2 \oslash a_1) = a_1$. Consequently, $a_2 \oslash a_1 = (a_2 \oslash a_1) \wedge a = a \oslash (a \otimes (a_2 \oslash a_1)) = a \oslash a_1 = a_2$ whence $a_1 \wedge a_2 = a_2 \otimes (a_2 \oslash a_1) = 0$.

Now, for any $x \in X$, $x \wedge a_2 \in X$ and $x \wedge a_2 \leqslant a$, thus $x \wedge a_2 \leqslant a_1$, whence we obtain $x \wedge a_2 \leqslant a_1 \wedge a_2 = 0$. This proves that $a_2 \in X^\delta$.

Observe that a_2 is the greatest element of $X^\delta_a = \{y \in X^\delta : y \leqslant a\}$. Indeed, if $y \in X^\delta$ then $y \leqslant a$ implies $y = y \oslash a_1 \leqslant a \oslash a_1 = a_2$.

It remains to be shown that $a = a_1 \vee a_2$. Since $a_1, a_2 \leqslant a$, we can compute $a_1 \vee a_2$ in the pseudo-MV-algebra $[0, a]$: $a_1 \vee a_2 = a \otimes ((a \oslash a_1) \wedge (a \oslash a_2)) = a \otimes ((a \oslash a_1) \oslash (a_2 \oslash a_1)) = a \otimes (a_2 \oslash a_2) = a$. □

One readily sees that direct factors are closed under existing suprema, and hence a direct factor of an orthogonally σ-complete commutative pseudo-BCK-algebra is an orthogonally σ-complete commutative pseudo-BCK-algebra again.

Lemma 3.2 *Let A be an orthogonally σ-complete commutative pseudo-BCK-algebra. If X_1 is a direct factor in A, and X_2 is a direct factor in X_1, then X_2 is a direct factor in A. Likewise, if X_1, X_2 are direct factors in A such that $X_2 \subseteq X_1$, then X_2 is a direct factor in X_1.*

Proof Every $a \in A$ is of the form $a = x_1 \vee y_1$, where $x_1 \in X_1$ and $y_1 \in X_1^\delta$. Moreover, $x_1 \in X_1$ can be written as $x_1 = x_2 \vee y_2$ for some $x_2 \in X_2$ and $y_2 \in X_2^{\delta_1}$, where $X_2^{\delta_1} = \{u \in X_1 : u \wedge v = 0 \text{ for all } v \in X_2\}$ is the polar of X_2 in X_1. Hence $a = x_2 \vee y_2 \vee y_1$, where $x_2 \in X_2$ and $y_2 \vee y_1 \in X_2^\delta$. Indeed, since $y_2 \in X_2^{\delta_1} \subseteq X_2^\delta$ and $y_1 \in X_1^\delta \subseteq X_2^\delta$, we have $z \wedge (y_2 \vee y_1) = (z \wedge y_2) \vee (z \wedge y_1) = 0$ for each $z \in X_2$.

For the latter claim, if $a \in X_1$ then $a = x \vee y$ for some $x \in X_2$, $y \in X_2^\delta$. Since $y \leqslant a \in X_1$, also $y \in X_1$ and hence $y \in X_1 \cap X_2^\delta = X_2^{\delta_1}$. □

4 Proof of Theorem A

Lemma 4.1 *Let $(A, \oslash, \otimes, 0)$ be an orthogonally σ-complete commutative pseudo-BCK-algebra and $\{X_n : n \in \mathbb{N}\}$ a countable family of direct factors in A so that $X_i \cap X_j = \{0\}$ for all $i, j \in \mathbb{N}$, $i \neq j$. Then*

$$X_0 := \bigcap_{n \in \mathbb{N}} X_n^\delta$$

is a direct factor in A. *Moreover,*

$$A \cong X_0 \times \prod_{n\in\mathbb{N}} X_n.$$

Proof For an arbitrary element $z \in A$ and $n \in \mathbb{N}$, let z_n denote the "X_n-coordinate" of z in the direct sum $X_n \oplus X_n^\delta = A$, i.e., z_n is the greatest element of X_n below z.

Let $a \in A$. For every $i, j \in \mathbb{N}$, $i \neq j$, we have $a_i \wedge a_j \in X_i \cap X_j = \{0\}$, thus $a_i \wedge a_j = 0$, which ensures the existence of

$$b := \bigvee_{n\in\mathbb{N}} a_n.$$

Moreover, $b_n = a_n$ for each $n \in \mathbb{N}$. Indeed, since $a \geqslant a_n$ for all $n \in \mathbb{N}$, it holds $a \geqslant b$ and so $a_n \geqslant b_n$ for all n. Conversely, from $b \geqslant a_n$ it follows $b_n \geqslant a_n$.

Put $c = a \oslash b$. For every $n \in \mathbb{N}$ we have $c_n = a_n \oslash b_n = a_n \oslash a_n = 0$, yielding $c \in \bigcap_{n\in\mathbb{N}} X_n^\delta = X_0$. Furthermore, if $x \in X_0$ then $x \wedge b = x \wedge \bigvee_{n\in\mathbb{N}} a_n = \bigvee_{n\in\mathbb{N}}(x \wedge a_n) = 0$ as $x \wedge a_n = 0$ for all n, and hence $b \in X_0^\delta$. Note that $b \wedge c = 0$, because $b \in X_0^\delta$ and $c \in X_0$.

We know that $b, c \leqslant a$. Hence in the pseudo-MV-algebra $[0, a]$ we have $c \oplus_a b = (a \oslash b) \oplus_a b = a \vee b = a$, and from $c \wedge b = 0$ it follows that $a = c \oplus_a b = c \vee b$.

Altogether, every $a \in A$ can be written in the form $a = b \vee c$, where $b \in X_0^\delta$ and $c \in X_0$, so by Lemma 3.1 X_0 is a direct factor in A.

By what we have established before, every $a \in A$ is of the form

$$a = c \vee \bigvee_{n\in\mathbb{N}} a_n,$$

where $c \in X_0$ and $a_n \in X_n$ for each $n \in \mathbb{N}$. To see that this expression is unique, suppose that $a = c' \vee \bigvee_{n\in\mathbb{N}} a'_n$ for some $c' \in X_0$ and $a'_n \in X_n$. Since $c, c' \in X_0$ and $\bigvee_{n\in\mathbb{N}} a_n, \bigvee_{n\in\mathbb{N}} a'_n \in X_0^\delta$, it follows $c' = c$ and $\bigvee_{n\in\mathbb{N}} a'_n = \bigvee_{n\in\mathbb{N}} a_n$, which yields (for every $k \in \mathbb{N}$) $a_k = a_k \wedge \bigvee_{n\in\mathbb{N}} a'_n = \bigvee_{n\in\mathbb{N}}(a_k \wedge a'_n) = a_k \wedge a'_k$ as $a_k \wedge a'_n = 0$ for $k \neq n$. Hence $a_k \leqslant a'_k$. Analogously, $a'_k \leqslant a_k$, and so $a'_k = a_k$ for all $k \in \mathbb{N}$.

Now we define $f\colon X_0 \times \prod_{n\in\mathbb{N}} X_n \to A$ by letting

$$f(x_n : n \in \mathbb{N}_0) := \bigvee_{n\in\mathbb{N}_0} x_n.$$

Obviously, f is a bijection. For any $(x_n : n \in \mathbb{N}_0), (y_n : n \in \mathbb{N}_0) \in X_0 \times \prod_{n\in\mathbb{N}} X_n$ we have $f(x_n : n \in \mathbb{N}_0) \oslash f(y_n : n \in \mathbb{N}_0) = (\bigvee_{n\in\mathbb{N}_0} x_n) \oslash (\bigvee_{n\in\mathbb{N}_0} y_n) = \bigvee_{n\in\mathbb{N}_0}(x_n \oslash (\bigvee_{k\in\mathbb{N}_0} y_k)) = \bigvee_{n\in\mathbb{N}_0} \bigwedge_{k\in\mathbb{N}_0}(x_n \oslash y_k) = \bigvee_{n\in\mathbb{N}_0}(x_n \oslash y_n) = f(x_n \oslash y_n : n \in \mathbb{N}_0) = f((x_n : n \in \mathbb{N}_0) \oslash (y_n : n \in \mathbb{N}_0))$ since for $n \neq k$, $x_n \wedge y_k = 0$ entails $x_n \oslash y_k = x_n$. Therefore f is an isomorphism. □

Lemma 4.2 *Let* $(A, \oslash, \owedge, 0)$ *be an orthogonally* σ*-complete commutative pseudo-BCK-algebra, and let* X_1, X_2 *be direct factors with* $X_2 \subseteq X_1$. *If* $A \cong X_2$ *then also* $A \cong X_1$.

Proof Let h be an isomorphism of A onto X_2. Define the sequence of deductive systems

$$X_0 := A, \qquad X_1, \qquad X_2, \qquad X_3 := h(X_1), \qquad X_4 := h(X_2), \qquad \text{etc.},$$

i.e., $X_{n+2} = h(X_n)$ for every $n \in \mathbb{N}_0$. It is clear that the restriction $h\restriction_{X_n}$ is an isomorphism of X_n onto X_{n+2}, so

$$A \cong X_2 \cong X_4 \cong \cdots \quad \text{and} \quad X_1 \cong X_3 \cong X_5 \cong \cdots$$

Hence we may assume $X_0 \supset X_1 \supset X_2 \supset \cdots$ since $X_n = X_{n+1}$ for some $n \in \mathbb{N}_0$ would yield $X_1 \cong A$.

Further, for any $n \in \mathbb{N}_0$, let $X_{n+1}^{\delta_n}$ denote the polar of X_{n+1} in X_n, i.e.,

$$X_{n+1}^{\delta_n} = \{x \in X_n : x \wedge y = 0 \text{ for all } y \in X_{n+1}\} = X_n \cap X_{n+1}^{\delta}.$$

We are going to show that

$$h(X_{n+1}^{\delta_n}) = X_{n+3}^{\delta_{n+2}}. \tag{4.1}$$

Let $x \in X_{n+3}^{\delta_{n+2}} = X_{n+2} \cap X_{n+3}^{\delta}$. Then $x = h(x_0)$ for some $x_0 \in X_n$, and for each $y_0 \in X_{n+1}$ we have $h(x_0 \wedge y_0) = h(x_0) \wedge h(y_0) = 0$ since $h(y_0) \in X_{n+3}$. Consequently, $x_0 \wedge y_0 = 0$, which yields $x_0 \in X_{n+1}^{\delta_n}$ and so $x \in h(X_{n+1}^{\delta_n})$. Conversely, let $x \in h(X_{n+1}^{\delta_n})$, i.e., $x = h(x_0)$ for some $x_0 \in X_{n+1}^{\delta_n} = X_n \cap X_{n+1}^{\delta}$. If $y \in X_{n+3}$ then $y = h(y_0)$ for some $y_0 \in X_{n+1}$, thus $x \wedge y = h(x_0 \wedge y_0) = 0$ since $x_0 \wedge y_0 = 0$. This means $x \in X_{n+2} \cap X_{n+3}^{\delta} = X_{n+3}^{\delta_{n+2}}$ which settles (4.1).

Next, we prove that

$$X_n = X_{n+1} \oplus X_{n+1}^{\delta_n},$$

in other words, X_{n+1} and $X_{n+1}^{\delta_n}$ are direct factors in X_n. By induction on $n \in \mathbb{N}_0$. For $n = 0$ this is just the hypothesis that X_1 is a direct factor in A, and for $n = 1$ it follows from this hypothesis by Lemma 3.2. Let $n \geqslant 2$ and suppose that the statement holds for all $k < n$. Then $X_{n-1} \oplus X_{n-1}^{\delta_{n-2}} = X_{n-2}$ and $X_n = h(X_{n-1} \oplus X_{n-1}^{\delta_{n-2}})$. Every $x \in X_n$ is therefore in the form $x = h(y \vee z) = h(y) \vee h(z)$, where $y \in X_{n-1}$ and $z \in X_{n-1}^{\delta_{n-2}}$. Since $h(y) \in X_{n+1}$ and $h(z) \in X_{n+1}^{\delta_n}$ by (4.1), it follows that $X_n = X_{n+1} \oplus X_{n+1}^{\delta_n}$.

Now, we put

$$Y_n := X_{n+1}^{\delta_n}.$$

Since $X_{n+1}^{\delta_n}$ is a direct factor in X_n, it is evident that all Y_n's are direct factors in A. Moreover, for $i \neq j$ we have $Y_i \cap Y_j = \{0\}$. Indeed, if e.g. $i < j$ then $X_i \supset X_j$, $X_{i+1}^{\delta} \subseteq X_{j+1}^{\delta}$ and $X_{i+1} \supseteq X_j$, whence we obtain $Y_i \cap Y_j = X_i \cap X_{i+1}^{\delta} \cap X_j \cap X_{j+1}^{\delta} = X_j \cap X_{i+1}^{\delta} \subseteq X_{i+1} \cap X_{i+1}^{\delta} = \{0\}$.

Hence, using Lemma 4.1,

$$A \cong Z \times \prod_{n \in \mathbb{N}_0} Y_n,$$

where $Z = \bigcap_{n \in \mathbb{N}_0} Y_n^{\delta}$. Obviously, for $n \geqslant 1$, $Y_n \subseteq X_1$ and $Z = Y_0^{\delta} \cap \bigcap_{n \in \mathbb{N}} Y_n^{\delta} = X_1^{\delta\delta} \cap \bigcap_{n \in \mathbb{N}} Y_n^{\delta} = X_1 \cap \bigcap_{n \in \mathbb{N}} Y_n^{\delta} = \bigcap_{n \in \mathbb{N}} (X_1 \cap Y_n^{\delta}) = \bigcap_{n \in \mathbb{N}} Y_n^{\delta_1}$, and therefore,

$$X_1 \cong Z \times \prod_{n \in \mathbb{N}} Y_n,$$

and since $Y_n \cong Y_{n+2}$ by (4.1), it follows

$$A \cong Z \times \prod_{n \in \mathbb{N}_0} Y_n \cong Z \times \prod_{n \in \mathbb{N}} Y_n \cong X_1.$$

□

We are ready to prove Theorem A, which can be reformulated as follows:

Theorem 4.3 *Let A and B be orthogonally σ-complete commutative pseudo-BCK-algebras. If A is isomorphic to a direct factor in B, and B is isomorphic to a direct factor in A, then $A \cong B$.*

Proof Let $f: A \to B$ and $g: B \to A$ be embeddings such that $f(A)$ and $A_1 := g(B)$ are direct factors in B and A, respectively. It suffices to observe that $A_2 := (f \circ g)(A)$ is a direct factor (in A_1 and hence) in A satisfying $A_2 \subseteq A_1$ and $A_2 \cong A$, which yields $A \cong A_1 \cong B$ by Lemma 4.2. □

5 Applications

Let A be a pseudo-MV-algebra (= bounded commutative pseudo-BCK-algebra). An element $e \in A$ is said to be *boolean* [4] if it has a complement in the underlying lattice of A. In this case, $e^- = e^\sim$ is the complement of e. This is also equivalent to $e \oplus e = e$. The boolean elements of A form a subalgebra that is a boolean algebra in its own right.

Following [8], we can say that $X \subseteq A$ is a direct factor if and only if there exists a boolean element $e \in A$ such that $X = [0, e]$. (This is also a corollary of our Lemma 3.1.) Therefore, by Example 2.2(b) and Theorem 4.3 we gain:

Corollary 5.1 [10] *Let A, B be orthogonally σ-complete pseudo-MV-algebras such that $A \cong [0, e] \subseteq B$ for some boolean element $e \in B$, and $B \cong [0, f] \subseteq A$ for some boolean element $f \in A$. Then $A \cong B$.*

This result is due to Jakubík [10] and extends the following MV-algebraic Cantor–Bernstein theorem which was proved by De Simone, Mundici and Navara [15] (it suffices to observe that every σ-complete MV-algebra is automatically an orthogonally σ-complete pseudo-MV-algebra):

Corollary 5.2 [15] *If A and B are two (orthogonally) σ-complete MV-algebras such that A is isomorphic to $[0, e] \subseteq B$ where e is a boolean element in B, and B is isomorphic to $[0, f] \subseteq A$ for some boolean element f in A, then $A \cong B$.*

It should be mentioned that there is another Cantor–Bernstein-like theorem for MV-algebras by Jakubík [7], but as observed in [15] it is incomparable with the above one.

Let us recall that an ℓ-group G is called *orthogonally* (or *laterally*) *σ-complete* if every pairwise orthogonal set of elements of G has a supremum in G. It is worth reminding that a convex ℓ-subgroup X of G is a direct factor if and only if for all $g \in G^+$, $X_g = \{x \in X : 0 \leqslant x \leqslant g\}$ has a greatest element.

Jakubík [9] proved the next theorem which can be easily achieved from Theorem A:

Corollary 5.3 [9] *Let G, H be orthogonally σ-complete ℓ-groups. If G is isomorphic to a direct factor in H, and H is isomorphic to a direct factor in G, then $G \cong H$.*

Proof Let $G \cong G_1$ and $H \cong H_1$, where G_1 and H_1 are direct factors in H and G, respectively. We may regard the positive cones G^+ and H^+ as orthogonally σ-complete commutative pseudo-BCK-algebras $(G^+, \oslash, \otimes, 0)$ and $(H^+, \oslash, \otimes, 0)$. The sets $G_1^+ = G_1 \cap H^+$ and $H_1^+ = H_1 \cap G^+$ are direct factors in H^+ and G^+, respectively, and it is plain that $G^+ \cong G_1^+$ and $H^+ \cong H_1^+$. By Theorem A (or 4.3), the pseudo-BCK-algebras G^+ and H^+ are isomorphic, and consequently, the ℓ-groups G and H are isomophic as well. □

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